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Left-right entanglement entropy of Dp-branes

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ABSTRACT: We compute the left-right entanglement entropy for Dp-branes in string theory. We employ the CFT approach to string theory Dp-branes, in particular, its presentation as coherent states of the closed string sector. The entanglement entropy is computed as the von Neumann entropy for a density matrix resulting from integration over the left-moving degrees of freedom. We discuss various crucial ambiguities related to sums over spin structures and argue that different choices capture different physics; however, we advance a thermodynamic argument that seems to favor a particular choice of replica. We also consider Dp branes on compact dimensions and verify that the effects of T-duality act covariantly on the Dp brane entanglement entropy. We find that generically the left-right entanglement entropy provides a suitable generalization of boundary entropy and of the D-brane tension.

KEYWORDS: D-branes, Conformal Field Models in String Theory

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1 Introduction

String theory is the leading candidate for a theory of quantum gravity. In that role one of its most impressive results is a microscopic explanation for the Bekenstein-Hawking entropy of certain black holes [1, 2]. In this context, and in further descriptions of black hole dynamics including low energy emission, D-branes play a central role.

Dp-branes can be viewed as coherent states in the corresponding string theory. In this manuscript, we explore some of its aspects from this point of view, in the hope of ultimately clarifying their role in the context of black holes. The characteristic that we focus on is entanglement entropy. A natural motivation for the study of entanglement entropy in Dp-branes comes from the original considerations of entanglement entropy and its potential relation to black hole entropy [3, 4]. More generally, the entanglement entropy of excited states and, particularly, of coherent states has been recently discussed in the literature. Various results have been obtained in the context of condensed matter [5], including also applications to the entanglement of topologically ordered states [6].

One entanglement quantity, the left-right entanglement entropy (LREE) of coherent states was discussed in [7]. Some interesting generalizations of the LREE include its formulation in the more general context of arbitrary rational CFT [8], Ising model with interfaces [9], connection with level/rank duality [10]. More recent work [11] provided an alternative perspective on [8]; other interesting directions were discussed in [12].

An important motivation for our work comes from the hope of constructing a model similar to Maldacena's eternal black hole [13]. This model, albeit in higher dimensions, describe two coupled CFT's and focuses on the maximally entangled states. Ultimately, we hope to connect the left-right entanglement entropy with the thermal field theory entropy which might have implications for a potential resolution to the information paradox. There is, indeed, a precedent in the program of connecting entanglement entropy with thermal behavior and aspects of black hole physics as presented in [14].

In this manuscript, we take a step further in the general considerations of LREE and address Dp-branes states in string theory. From a technical point of view, a string theory is a CFT with added constraints. Since we work in the context of superstring theory we find various ambiguities in the choice and treatment of the various sectors. In this paper, we provide a complete string-theoretic calculation of the left-right entanglement entropy of Dp branes.

The rest of the paper is organized as follows. In section 2 we review the construction of D branes as boundary states in the closed string sector. In section 3 we introduce details of the density matrix and our prescriptions for implementing the replica trick. Section 4 presents the left-right entanglement entropy of Dp branes. Section 5 considers the effect of the normalization while replicating the theory. In section 6 we advance an argument, based on a potential thermodynamical interpretation, according to which a particular choice of correlations during the replica process is selected. We consider the case of compact spaces and the action of T-duality in section 7. We conclude and propose some open questions in section 8. We relegate to a series of appendices some technical material regarding properties of the theta functions and some explicit computations on the density matrix.

2 Dp branes in string theory

In this section we briefly review the relevant concepts of boundary states describing Dp-branes. A detailed exposition of the subject is given in [15] from where we use most of our notation. The preponderant geometrical view of a Dp-brane is that of a hyperplane where open strings can end. In this context Dp-branes are boundaries for the end points of the string. In the boundary conformal field theory approach, D-branes are boundary states of the closed string theory emitting or absorbing closed string states. A boundary state is constructed by solving specific equations relating left- and right-modes of closed strings at the boundary. We work in the light-cone gauge with light-cone directions x^0 and x^1 which are taken to be Dirichlet directions. In these conventions $-1 \leq p \leq 7$. Therefore, the boundary states actually will be describing D-instantons. However, one can perform an appropriate Wick rotation to transform these states back into usual D-branes. For the

transverse coordinates x^μ , $\mu = 2, \dots, 9$ one can further split the index as $\mu = (\alpha, i)$ with $\alpha = 2, \dots, p+2$ the Neumann directions and $i = p+3, \dots, 9$ the Dirichlet directions.

The solution, using the notation of [16], is:

$$|\mathbf{k}, \eta\rangle_{\text{RR}}^{\text{NSNS}} = \exp \left\{ \sum_{n>0} \left[\frac{1}{n} \alpha_{-n}^\mu S_{\mu\nu} \tilde{\alpha}_{-n}^\nu \right] + i\eta \sum_{r>0} \left[\psi_{-r}^\mu S_{\mu\nu} \tilde{\psi}_{-r}^\nu \right] \right\} |\mathbf{k}, \eta\rangle_{\text{RR}}^0, \quad (2.1)$$

where α_n denotes the bosonic operator with n an integer and ψ_r is the fermionic operator and depending on the sector, the mode number r is integer in R-R sector or half-integer in NS-NS sector. The matrix $S_{\mu\nu} = (\eta_{\alpha\beta}, -\delta_{i,j})$ denotes the Neumann or Dirichlet boundary conditions of the Dp-brane. The boundary ground state $|\mathbf{k}, \eta\rangle_{\text{NSNS}}^0$ is the unique ground state of the NS-NS sector of the bulk theory and it carries momentum \mathbf{k} in the Dirichlet directions only. The definition of the boundary ground state in the R-R sector is more involved but as we will explain later, we will not be interested in the R-R sector of the boundary states. For an explicit derivation of R-R ground state sector we refer the reader to [15]. The parameter $\eta = \pm$ denotes the two different spin structures that will play a prominent role in our story.

In order to localize the boundary state (2.1) one has to take its Fourier transform:

$$|Bp, \mathbf{a}, \eta\rangle_{\text{RR}}^{\text{NSNS}} = \mathcal{N} \int \prod_{\nu=0,1,p+3,\dots,9} dk^\nu e^{ik^\nu a_\nu} |\mathbf{k}, \eta\rangle_{\text{RR}}^{\text{NSNS}}, \quad (2.2)$$

where \mathbf{a} denotes the position of the boundary state, \mathcal{N} a normalization constant and the integration is over the Dirichlet (transverse) directions to the boundary. In the following we consider the case $\mathbf{a} = 0$ without loss of generality and in such case the boundary state is denoted simply as $|Bp, \eta\rangle$.

The boundary states (2.2) by themselves do not represent a Dp-brane. One has to further impose certain conditions based on physical grounds. One of the conditions required is GSO invariance of the boundary states, it ensures that the boundary states couple only to the physical spectrum of the closed string theory. This condition implies that physical boundary states are linear combinations of the boundary states (2.2). For the NS-NS sector, the GSO-invariant state is

$$|Bp\rangle_{\text{NSNS}} = \sum_{\eta=\pm} \eta |Bp, \eta\rangle_{\text{NSNS}}, \quad (2.3)$$

and the GSO invariant state for the R-R sector is

$$|Bp\rangle_{\text{RR}} = \sum_{\eta=\pm} |Bp, \eta\rangle_{\text{RR}}. \quad (2.4)$$

Another important condition to be imposed is the open-closed duality where the tree-level amplitude of a boundary state with itself, describing the exchange of closed string states between them, has to be equivalent, after an appropriate conformal transformation, to the one-loop open string amplitude. From the CFT point of view, this is the Cardy condition. This requirement guarantees that the open strings obtained by this duality have consistent interactions with the original closed strings.

This last condition allows to construct the D-branes as linear combinations of physical boundary states from different sectors. In Type IIA/B in ten dimensions there are two possibilities. The BPS brane

$$|Dp\rangle = |Bp\rangle_{\text{NSNS}} + |Bp\rangle_{\text{RR}} , \quad (2.5)$$

where p is even (odd) in Type IIA (IIB). This is a stable brane and the open string spectrum of the open strings beginning and ending on two D-branes is supersymmetric.

There is also a non-BPS brane defined as

$$|Dp\rangle = |Bp\rangle_{\text{NSNS}} , \quad (2.6)$$

for p odd (even) in Type IIA (IIB). This is an unstable brane and the spectrum of open strings is never supersymmetric.

3 Density matrix for D-branes

Given a system in a pure state $|\psi\rangle$, it can be characterized by the density operator $\rho = |\psi\rangle\langle\psi|$ acting on the state space \mathcal{H} . Since the system is in a pure state, the density matrix is Hermitian and satisfy $\rho^2 = \rho$ and $\text{Tr}\rho = 1$. If the system is factorized into two different subsystems A and B with Hilbert space the tensor product $\mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_B$, one can construct a reduced density matrix from ρ describing the physical properties of one of the subsystems. For instance, the reduced density matrix of subsystem A is obtained by taking the partial trace of ρ with respect to subsystem B, that is, $\rho_A = \text{Tr}_B \rho$. At this point, ρ_A allows one to describe the system A as if it were effectively isolated from system B. However, systems A and B could be entangled showing quantum correlations between the two systems. There are several ways to measure entanglement. In this work we are interested in the entanglement entropy for subsystem A, which is defined as the von Neumann entropy of the reduced density matrix ρ_A :

$$S_A = -\text{Tr} \rho_A \log \rho_A, \quad (3.1)$$

a similar definition follows for S_B , moreover, $S_A = S_B$ for a system in a pure state. One efficient way to arrive at the entanglement entropy above is through the Rényi entropies

$$S^{(n)}(\rho_A) = \frac{1}{1-n} \log \text{Tr} \rho_A^n = S^{(n)}(\rho_B),$$

where $n \geq 0$. The entanglement entropy is then defined as $S_A = \lim_{n \rightarrow 1} S_A^{(n)}$. A convenient way to compute $\text{Tr} \rho_A^n$ for general real n is to employ the replica trick. This prescription consists in computing $\text{Tr} \rho_A^n$ for $n \in \mathbb{Z}^+$ and then taking the analytic continuation to $\text{Re } n > 1$. By this approach one obtains

$$\text{Tr} \rho_A^n = \frac{Z_n(A)}{Z^n}, \quad (3.2)$$

where $Z_n(A)$ is a partition function on a n -sheeted Riemann surface.

Typically, subsystems A and B are defined as geometric regions of the whole system. In this paper, as in our previous one [7], we will deviate from this geometric prescription. Our whole system is an extended object in the supersymmetric closed type IIA/B string theory. Excitations of the closed string can be decomposed into a superposition of modes that travel clockwise (left-moving) around the string and those that travel counter-clockwise (right-moving). Therefore in this work we define the subsystems A and B as the left- and right-moving modes of closed superstring theories, respectively. In particular, we are interested in computing entanglement entropy of left and right modes of boundary states defining Dp-branes in string theory.

Before discussing the reduced density matrix for the subsystem A we note that in order to define the density matrix for the branes we have to introduce a regularization factor $e^{-\epsilon H}$ due to the fact that boundary states are non-normalizable. An extensive discussion of the meaning of this regularization is presented in [7] where some simple computations making explicit the physical interpretation of ϵ were carried out. In this case H is the Hamiltonian of the supersymmetric closed string and the density matrix for the boundary state in the NS-NS sector or R-R sector is given as

$$\rho^{\text{NS}}_{\text{R}} = \frac{1}{Z} e^{-\epsilon H} |Bp\rangle_{\text{RR}}^{\text{NSNS}} \langle Bp| e^{-\epsilon H} , \quad (3.3)$$

with $|Bp\rangle_{\text{RR}}^{\text{NSNS}}$ are defined by eq. (2.3) and eq. (2.4). Note that one can consider the brane to be localized in \mathbf{a} as in eq. (2.2), however, the term with the position of the brane does not contribute to trace of ρ since the adjoint brane is also taken to be localized at the same place. Therefore, we will consider branes localized at the origin. For the BPS brane (2.5) the density operator is $\rho = \rho^{\text{NS}} \oplus \rho^{\text{R}}$. A geometrical interpretation of ρ is presented in figure 1 where external closed string states propagate a time ϵ , they are absorbed by a brane and after that, they are re-emitted by the same brane propagating a time ϵ . Note that the circumference of the cylinder displayed by the closed string state has been settled, to one, as is usual.

Let us denote the closed string states as $|l, \tilde{l}, \mathbf{k}\rangle$. When taking the matrix elements $\langle \mathbf{k}, l, \tilde{l} | \rho | l', \tilde{l}', \mathbf{k}' \rangle$ the trace imposes the periodic boundary conditions $|l', \tilde{l}', \mathbf{k}'\rangle = |l, \tilde{l}, \mathbf{k}\rangle$. After taking the trace over the closed string states and imposing the condition $\text{Tr} \rho = 1$ one finds that

$$Z = \mathcal{N}^2 \left(\frac{1}{2\epsilon} \right)^{\frac{9-p}{2}} \frac{f_3^8(q) - f_4^8(q) - f_2^8(q)}{f_1^8(q)} = \langle Dp | e^{-2\epsilon H} | Dp \rangle . \quad (3.4)$$

The result of tracing is given in the first equality, this term is the same as the interaction force between two BPS-branes given by the second equality. The geometry after taking the trace over the closed strings is a cylinder with boundaries as in figure 1 representing the interaction between the branes exchanging closed string states propagating a time 2ϵ between the branes. The functions f_i for $i = 1, 2, 3, 4$ are defined in appendix A with $q = e^{-4\pi\epsilon}$ ($\tau = 4i\epsilon$). The partition function Z so defined is identically zero (by Jacobi's abstruse identity) revealing the fact that the exchange of NS-NS and R-R states cancel. Therefore, in this case the density matrix (3.3) is ill defined, since even after the

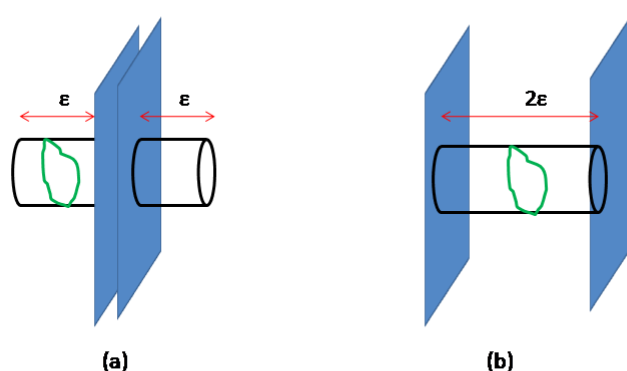


Figure 1. The geometrical representation of the unnormalized ρ_{NSNS} is given in (a). The length of each cylinder is ϵ with circumference 1. The figure (b) represents the interaction of two branes emitting and absorbing closed string states in a time 2ϵ , the circumference of the cylinder is also 1.

regularization $e^{-\epsilon H}$ we are unable to normalize the correspondent density matrix. At this conceptual junction, forced upon us technically by Jacobi's abstruse identity, we take inspiration in Polchinski's computation of D-brane tension and choose to focus only on the contribution of the boundary state in the NS-NS sector of the brane. Therefore, we will take in our discussions only the NS-NS sector. In such case, the density matrix for the BPS and non-BPS brane, is just $\rho = \rho^{\text{NS}}$ and it is well defined as we will see shortly.

In order to construct the reduced density matrix, we note that the space of states in the NS-NS sector of the closed superstring theory can be expressed as $\mathcal{H}_{\text{NS}} = \mathcal{H}_{\text{NS}}^L \otimes \mathcal{H}_{\text{NS}}^R$. An orthogonal basis for the left (L) mode space is

$$|l_L, \mathbf{k}\rangle = \prod_{\mu=2}^9 \prod_{n=0}^{\infty} \frac{1}{\sqrt{l_n!}} \left(\frac{\alpha_{-n}^{\mu}}{\sqrt{n}} \right)^{l_n} \prod_r \left(\psi_{-r}^{\mu_r} \right)^{l_r} |0, \mathbf{k}\rangle, \quad (3.5)$$

with $|0, \mathbf{k}\rangle$ the NS-ground state. A similar relation is given for the right (R) modes. Although the oscillator modes can be separated as left- and right modes, the closed string ground state carries only one momentum \mathbf{k} .

The boundary state given in (2.2) can be expressed in terms of the orthonormal basis of the space of states \mathcal{H}_{NS} . In fact, the boundary state is not a product state on such space but an entangled state in the right- and left-modes. Then we can see the brane as a bipartite system consisting of subsystems A (right-modes) and B (left-modes). To measure the entanglement we define the reduced density matrix for system A. Since the NS-NS ground state is labeled by the momentum \mathbf{k} we have to sum also over the momenta. We define the reduced density matrix for A as:

$$\rho_A = \text{Tr}_B(\text{Tr}_C \rho), \quad (3.6)$$

where we have denoted the momentum by C. Similarly $\rho_B = \text{Tr}_A(\text{Tr}_C \rho)$ and $\rho_C = \text{Tr}_A(\text{Tr}_B \rho)$. For the bipartite entanglement, $S_A = S_B > 0$ but $S_C = 0$.

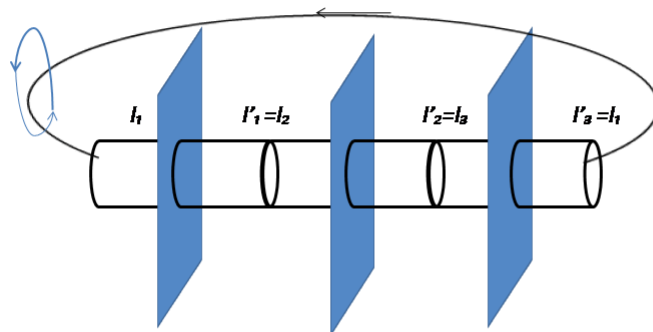


Figure 2. This graph represents a right-mode closed string propagating between three different branes for $\text{Tr}\rho_A^3$.

In order to simplify our notation we will drop the labels of the states and wherever we write $|Bp\rangle$ and ρ we mean $|Bp\rangle_{\text{NSNS}}$ and ρ^{NS} . The condition $\text{Tr}\rho = 1$ fixes the normalization constant as

$$Z = \text{Tr}\langle\langle\rho\rangle\rangle, \quad (3.7)$$

where we use the notation $\langle\langle\ldots\rangle\rangle$ to denote un-normalized density matrices [15]. The trace is performed on \mathcal{H}_{NS} and we obtain:

$$Z = 2\mathcal{N}^2 \left(\frac{1}{2\epsilon} \right)^{\frac{9-p}{2}} \frac{f_3^8(q) - f_4^8(q)}{f_1^8(q)}. \quad (3.8)$$

Note that this equation is similar to equation (3.4) except for the contribution of the RR-sector given by $f_2(q)$ term. As in the supersymmetric case, this expression can be interpreted as a closed string state in the NS-NS sector, emitted from a NS-NS boundary state, the closed string state propagates for a time 2ϵ and then is absorbed by the same boundary state. Therefore, this normalization constant is the partition function describing the interaction of two boundary states exchanging NS-NS closed string states

$$Z = \langle Bp | e^{-2\epsilon H} | Bp \rangle, \quad (3.9)$$

with $|Bp\rangle$ defined as in (2.3).

4 Correlated and uncorrelated replica tricks

We are interested in computing the replicated partition function $Z_n = \text{Tr}\langle\langle\rho_A^n\rangle\rangle$. When computing this trace, the right-mode oscillators satisfy the conditions $l'_i = l_{i+1}$ and $l_1 = l'_n$ for $i = 1, 2, \dots, n$. A geometric representation of this process is given in figure 2 (for $n = 3$).

Note that our fermionic part of the reduced density matrix will contain a sum over the spin structures which is inherited from the fermionic part of the boundary state. Here, we consider two possibilities to carry out the replica trick on the fermions according to [17]. One can consider other possibilities but we will study only the “extreme” cases. In the first one, which is called the *uncorrelated spin structure*, the spin structures are summed over before the replication as is shown in figure 3(a). In this form, the spin structure of the copy denoted by $n = 1$ is disconnected with the spin structure of the next copy $n = 2$ and so on.

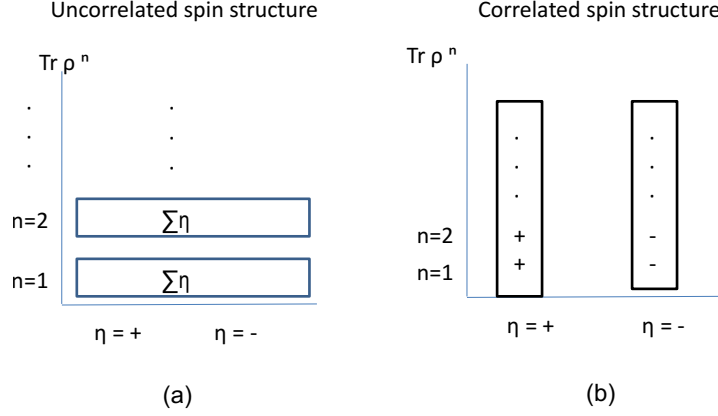


Figure 3. A graphic representation of the choices in replicating the theory as related to the treatment of spin structures.

The other extreme possibility for replicating is called the *correlated spin structure*. The spin structures in the fermionic part of the reduced density matrix are separated and the replica is performed over each spin structure η separately. After the replication, the spin structures are summed. This case is represented in figure 3(b).

In addition to the spin structures, the boundary states are also defined by the momentum. Due to the fact that one localizes the brane by the Fourier transform given in eq. (2.1), integration over momentum follows. Therefore, we consider also two possibilities for replicating depending on whether one integrates over the momentum before of after replicating. On the *uncorrelated momentum*, one integrates the momentum on each replica. For the *correlated momentum* one integrates over momentum after replicating.

4.1 Correlated and uncorrelated spin structure with uncorrelated momenta

Let us first describe the case of uncorrelated spin structure. The reduced density matrix, $\langle\langle\rho_A\rangle\rangle$ for system A as given in eq. (B.9). Note that the fermionic part of this unnormalized reduced density matrix is of the form

$$\langle\langle\rho_A\rangle\rangle_f = 2 \sum_{\eta} \prod_{\mu=1}^8 \prod_r \eta \left[1 \otimes 1 + \eta q^{2r} \psi_{-r}^{\mu} \otimes \psi_r^{\mu} \right] |0\rangle \otimes \langle 0|. \quad (4.1)$$

The factor of 2 comes from the fact that the product $\eta\eta'$ in (B.9) contributes twice to positive sign and twice to negative sign. When replicating over the system A, we have to replicate this sum. In this way, at each step of the replica the spin structures have been summed and, therefore, they are uncorrelated between the n -copies of $\langle\langle\rho_A\rangle\rangle$ as shown in figure 3(a).

The partition function in this case is called the uncorrelated replica partition function, denoted by Z_n^u and is given by

$$Z_n^u = 2^{n-1} 2^n \mathcal{N}^{2n} \left(\frac{1}{2\epsilon} \right)^{\left(\frac{9-p}{2}\right)n} \left(\frac{f_3^8(q_n) - f_4^8(q_n)}{f_1^8(q_n)} \right), \quad (4.2)$$

where $q_n = e^{i\pi\tau_n}$ with $\tau_n = 4i\epsilon n$. The factor 2^{n-1} results from the approach we are taking, since we are raising a sum over spin structures to a certain power. Note that as $\epsilon \rightarrow 0$, we have $q_n \rightarrow 1$, therefore, in order to perform an expansion for small ϵ we have to make use of the transformation $\tau_n \rightarrow -1/\tau_n$. In this way $q_n = e^{-4\pi\epsilon n}$ transforms as $\tilde{q}_n = e^{-\frac{\pi}{4\epsilon n}}$ and the transformation of the functions f_i are given in eq. (A.2). Therefore,

$$\frac{Z_n^u}{Z^n} \rightarrow 2^{n-1} \frac{(4\epsilon n)^4}{(4\epsilon)^{4n}} \left(\frac{f_1^8(\tilde{q})}{f_1^8(\tilde{q}) - f_2^8(\tilde{q})} \right)^n \left(\frac{f_3^8(\tilde{q}_n) - f_2^8(\tilde{q}_n)}{f_1^8(\tilde{q}_n)} \right). \quad (4.3)$$

As $\epsilon \rightarrow 0$, \tilde{q} goes to zero and we can use eq. (A.1) to expand the functions f_i for $i = 1, 2, 3$ and get the leading terms

$$\frac{Z_n^u}{Z^n} \rightarrow 2^{n-1} e^{-\frac{\pi}{4\epsilon}(n-\frac{1}{n})}, \quad (4.4)$$

yielding an entanglement entropy of the form:

$$S^u = \lim_{n \rightarrow 1} \frac{1}{1-n} \log \text{Tr} \frac{Z_n^u}{Z^n} = -\ln 2 + \frac{\pi}{2\epsilon}. \quad (4.5)$$

Next we are going to analyze the correlated replica trick. Starting from (4.1) we separate the spin structures and we write

$$\langle \langle \rho_A \rangle \rangle_f = 2(\rho_+ - \rho_-), \quad (4.6)$$

where $\rho_{+/-}$ denotes the term in (4.1) with positive/negative spin structure.

We then take the replica trick for each spin structure as in figure 3(b). After that, we sum over the spin structures. The partition function obtained in this way is called the correlated partition function and is denoted as Z_n^c . It has the form

$$Z_n^c = 2\mathcal{N}^{2n} \left(\frac{1}{2\epsilon} \right)^{\left(\frac{9-p}{2}\right)n} \left(\frac{f_3^8(q_n) - f_4^8(q_n)}{f_1^8(q_n)} \right). \quad (4.7)$$

Note that the factor of 2 in (4.1) is not replicated. As in the previous case, after the transformation $\tau_n \rightarrow -1/\tau_n$ and considering the limit $\epsilon \rightarrow 0$ we get:

$$S^c = \lim_{n \rightarrow 1} \frac{1}{1-n} \log \text{Tr} \frac{Z_n^c}{Z^n} = \ln 2 + \frac{\pi}{2\epsilon}. \quad (4.8)$$

In both cases, the momentum was already summed before the replica and the replication was made over system A only. Therefore, each replica has uncorrelated momenta. Note that the normalization constant has been replicated in both cases. Therefore, the normalization constant as well as the momentum contribution to Z_n^u and Z_n^c cancel with those terms in Z^n .

4.2 Uncorrelated spin structure with correlated momentum

One could also consider the reduced density matrix $\langle \langle \rho_{AC} \rangle \rangle = \text{Tr}_B \rho$. This expression is given in eq. (B.10), where the trace is made over the left-modes only. We then perform the replication for this matrix. Only after constructing such matrix we sum over the

momentum. In this case all the n -copies have the same momentum. This is what we call the correlated momentum. In addition we have to consider the two forms to carry out the replica trick for the spin structures.

We denote the replica partition function as $Z'_n = \text{Tr} \langle \langle \rho_{AC}^n \rangle \rangle$. When taking the correlated momenta and uncorrelated spin structure the partition function is:

$$Z'^u_n = 2^{n-1} 2^n \mathcal{N}^{2n} \left(\frac{1}{2\epsilon n} \right)^{\left(\frac{9-p}{2}\right)} \left(\frac{f_3^8(q_n) - f_4^8(q_n)}{f_1^8(q_n)} \right). \quad (4.9)$$

Note that the momentum contribution to this expression differs from that in (3.8). Therefore, this term is not canceled with the momentum contribution of Z^n . In such case the entanglement entropy becomes

$$S'^u = -\log 2 + \frac{9-p}{2} \log 2 + \frac{\pi}{2\epsilon}. \quad (4.10)$$

For correlated momentum an correlated spin structure over the replicas

$$Z'^c_n = 2 \mathcal{N}^{2n} \left(\frac{1}{2\epsilon n} \right)^{\left(\frac{9-p}{2}\right)} \left(\frac{f_3^8(q_n) - f_4^8(q_n)}{f_1^8(q_n)} \right). \quad (4.11)$$

The corresponding entanglement entropy becomes:

$$S'^c = \log 2 + \frac{9-p}{2} \log 2 + \frac{\pi}{2\epsilon}. \quad (4.12)$$

We note that all the previous replica partition function (4.7), (4.2), (4.9) and (4.11) reduce to (3.8) for $n = 1$:

$$Z_{n=1}^c = Z_{n=1}^u = Z'^c_{n=1} = Z'^u_{n=1} = Z. \quad (4.13)$$

5 Unreplicated normalization

The partition function Z given in eq. (3.8) is not invariant under the transformation $\tau \rightarrow -1/\tau$ since we are dealing with the NS-NS sector only. However, we noted that Z can be expressed as $\langle B | e^{-\epsilon H} | B \rangle$ which is the propagation amplitude between the same NS-NS boundary state. Such partition function satisfy, up to integrand factors the open/closed string duality. In [15] this duality has been described. For instance, for our NS-NS brane it was found that

$$\int_0^\infty dl \langle Bp, \eta | e^{-lH_c} | Bp, \eta \rangle = \mathcal{N}^2 \int_0^\infty dl \left(\frac{1}{l} \right)^{\frac{9-p}{2}} \frac{f_3^8(q)}{f_1^8(q)} \quad (5.1)$$

$$= \mathcal{N}^2 \frac{32(2\pi)^{p+1}}{V_{p+1}} \int_0^\infty \frac{dt}{2t} \text{Tr}_{\text{NS}} \left[e^{-tH_o} \right], \quad (5.2)$$

$$\int_0^\infty dl \langle Bp, \eta | e^{-lH_c} | Bp, -\eta \rangle = \mathcal{N}^2 \int_0^\infty dl \left(\frac{1}{l} \right)^{\frac{9-p}{2}} \frac{f_3^8(q)}{f_1^8(q)} \quad (5.3)$$

$$= \mathcal{N}^2 \frac{32(2\pi)^{p+1}}{V_{p+1}} \int_0^\infty \frac{dt}{2t} \text{Tr}_R \left[e^{-tH_o} \right], \quad (5.4)$$

where l is the length of the cylinder in the closed string channel and t the circumference of the cylinder in the open string channel. These relations fix the normalization constant for the non-BPS brane (2.6) as

$$\mathcal{N}_{\text{nonBPS}}^2 = \frac{V_{p+1}}{64(2\pi)^{p+1}}, \quad (5.5)$$

and for the BPS brane (2.5) the NS-NS normalization constant as

$$\mathcal{N}_{\text{BPS}}^2 = \frac{V_{p+1}}{128(2\pi)^{p+1}}. \quad (5.6)$$

We can ask then, if some of the replicated partition functions Z_n presented in section 3 satisfy the closed-open string duality. In all cases above, we have considered the natural way of computing the trace, taking the replication of the normalization constant giving rise to the factor \mathcal{N}^{2n} , therefore, none of the replicated partition functions (4.2), (4.7), (4.9) and (4.11) satisfy this duality. However, we can take another approach for the replication Z_n . Following [17] we redefine Z_n multiplying it by a term independent of ϵ , that is:

$$\tilde{Z}_n = \mathcal{N}^{2(1-n)} Z_n. \quad (5.7)$$

This is equivalent to requiring the normalization constant \mathcal{N} not to be replicated. From all the replicated partition function Z_n , correlated and uncorrelated, only Z'_n with the redefinition given above, can be related to the closed-open string duality. Then, the partition function with uncorrelated normalization and correlated momentum and spin structure has the form:

$$\tilde{Z}'^c_n = 2\mathcal{N}^2 \left(\frac{1}{2\epsilon n} \right)^{\left(\frac{9-p}{2}\right)} \left(\frac{f_3^8(q_n) - f_4^8(q_n)}{f_1^8(q_n)} \right), \quad (5.8)$$

which can be expressed as

$$\tilde{Z}'^c_n = \langle Bp | e^{-2\epsilon n H} | Bp \rangle. \quad (5.9)$$

This represents the amplitude for NS-NS closed string states propagating between the same brane. Such expression satisfies the open-closed duality under which the closed string cylinder of circumference 1 and length $2\epsilon n$ is changed to a cylinder in the open string channel with length 1 and circumference $1/4\epsilon n$. The normalization constant does not depend on the geometry of the cylinder. In the limit $\epsilon \rightarrow 0$, the leading terms in the trace are given by the lightest open string states. In this case we have:

$$\frac{\tilde{Z}'^c_n}{Z^n} = 2^{1-n} \mathcal{N}^{2(1-n)} \left(\frac{1}{2\epsilon n} \right)^{\frac{9-p}{2}} \left(\frac{1}{2\epsilon} \right)^{n\frac{p-9}{2}} \left(\frac{f_1^8(\tilde{q})}{f_3^8(\tilde{q}) - f_2^8(\tilde{q})} \right)^n \left(\frac{f_3^8(\tilde{q}_n) - f_2^8(\tilde{q}_n)}{f_1^8(\tilde{q}_n)} \right) \frac{(4\epsilon n)^4}{(4\epsilon)^{4n}}. \quad (5.10)$$

The entanglement entropy for the non-BPS brane (2.6) is then

$$\tilde{S}'^c = \log 2 + \log \frac{V_{p+1}}{64(2\pi)^{p+1}} + \frac{9-p}{2} \log 2 + \frac{\pi}{2\epsilon}. \quad (5.11)$$

We note that this expression depends on the normalization of the brane (related to the tension of the brane). In [18] an expression for the tension and the boundary entropy

was found. Here, we observe that the term of logarithm of the volume is precisely the boundary entropy of the brane, given as $\langle 0|Bp\rangle$ [19]. In this way we found a relation between the left-right entanglement entropy and the boundary entropy of the brane. After simplifications (5.11) reduces to

$$\tilde{S}'^c = \log \frac{V_{p+1}}{(2\pi)^{p+1}} - \frac{p+1}{2} \log 2 + \frac{\pi}{2\epsilon}. \quad (5.12)$$

Similarly, for the BPS brane, the normalization constant is one half the normalization of the non-BPS brane due to the GSO projection, therefore the entanglement entropy is

$$\tilde{S}'^c_n = \log \frac{V_{p+1}}{2(2\pi)^{p+1}} - \frac{p+1}{2} \log 2 + \frac{\pi}{2\epsilon}. \quad (5.13)$$

6 A potential thermodynamic interpretation

Given the ambiguities that permeate the definition of entanglement entropy, it makes sense to look for additional criteria that might help us choose a particular prescription over others. With this aim we would like to consider a structure akin to a thermodynamic limit. So far, the string theory and the boundary states have been defined at zero temperature. However, we can try to relate (in a fictitious way) the factor 2ϵ in (5.9) with the temperature. In such case, the high-temperature limit of equation (5.10) is dominated by the lowest energy states, that is, the massless closed string modes:

$$\text{Tr} \frac{\tilde{Z}'^c_n}{Z^n} \sim \mathcal{N}^{2(1-n)} \left(\frac{1}{2\epsilon n} \right)^{\frac{9-p}{2}} \left(\frac{1}{2\epsilon} \right)^{n \frac{p-9}{2}} 16^{1-n}.$$

For this case the left-right entanglement entropy is:

$$S_A(\beta) = \lim_{n \rightarrow 1} \frac{1}{1-n} \text{Tr} \frac{\tilde{Z}'^c_n}{Z^n} = 5 \log 2 + \log \mathcal{N}^2 + \left(\frac{9-p}{2} \right) \log \beta - \frac{9-p}{2}. \quad (6.1)$$

Our key observation is that this result agrees with the thermodynamical entropy computed from the partition function (3.8) in the limit $\beta = 2\epsilon \rightarrow 0$

$$S_{th} = \beta^2 \frac{\partial}{\partial \beta} \left(-\frac{1}{\beta} \log Z \right) = S_A(\beta). \quad (6.2)$$

A number of comments are in order. First, according to this criterion, it seems that the uncorrelated normalization and correlated momentum and spin structure entanglement entropy is the one that is favored. We highlight, however, that this criterion is just that, a way to concoct an physically reasonable extra condition that might help us understand which of the various procedures (correlated or uncorrelated replicas) leads to a more meaningful result.

Second, the high temperature limit is, in a sense a fairly universal limit. Indeed, one expects that at very high temperatures the system will be guided by universal ideas of equipartition of energy. The crucial point here is that, due to the nature of string theory,

this high temperature limit essentially picks out the massless states of the theory. This is quite different from the standard particle picture where it is the low temperature limit that picks the states with the lowest energies.

Perhaps more speculatively, it is worth mentioning that there have been some other contexts where the entanglement entropy seems to be connected to a thermodynamic entropy as evidenced in first-law like relations discussed in [20, 21]. Such thermodynamic properties of the entanglement entropy were, in a sense, anticipated in particular cases like [22] where by considering varying the length of the entanglement region a confinement/deconfinement transition was suggested. This reasoning leads to the treatment, effectively of the length of the interval as an effective temperature. More importantly, this effective temperature, as discussed in, for example [23], is independent of the “temperature” of the dual supergravity background.

7 Compactification and T-duality

For the case of compactification on a torus T^k , the boundary states describing D-branes have some small differences from the uncompactified case. The part corresponding to the non-zero modes is the same as before but the part of the zero-modes is modified since they are now characterized by quantized momenta and winding number along the compact directions. Along the Dirichlet directions the boundary state carries momentum $\frac{m_i}{R_i}$ and is denoted by $|B, m_i\rangle$. The state $|B, w_i\rangle$ is the boundary state along the Neumann directions carrying winding number $w_i R$. In the localized boundary state the integral of momenta over compact directions is replaced by a sum over the momenta m_i . One also has to sum over the winding numbers. For this case, there are $r + 1$ Neumann directions along the non-compactified coordinates and s Neumann compactified directions, with $p = r + s$. The boundary state is:

$$|Bp, \mathbf{a}, \theta, \eta\rangle = \mathcal{N}^2 \int \prod_{\mu} dp^{\mu} e^{ip \cdot \mathbf{a}} \prod_{i=1}^s \sum_{w_i} e^{i\theta \cdot w_i R} \prod_{j=1}^{k-s} \sum_{m_j} e^{i\hat{a} \cdot \frac{m_j}{R}} |Bp, \mathbf{p}, \mathbf{m}, \mathbf{w}, \eta\rangle, \quad (7.1)$$

where $\mathbf{a} = (a_{\mu}, \hat{a}_j)$ denotes the position of the brane along the transverse non-compact and compact directions to the brane, respectively. The parameter θ is a Wilson line associated with the U(1) gauge field living on the Dp-brane.

As in the previous case $Z = \text{Tr}\langle\langle \rho_A^n \rangle\rangle$ is the amplitude between two coincident branes $\langle Bp, \theta, \eta | e^{-\epsilon H} | Bp, \theta, \eta \rangle$:

$$Z = 2\mathcal{N}^2 \left(\frac{1}{2\epsilon} \right)^{\frac{9-(r+k)}{2}} \prod_{i=1}^s \sum_{w_j} e^{-2\pi\epsilon(w_i R_i)^2} \prod_{j=1}^{k-s} \sum_{m_j} e^{-2\pi\epsilon \left(\frac{m_j}{R_j} \right)^2} \frac{f_3^8(q) - f_4^8(q)}{f_1^8(q)}. \quad (7.2)$$

Following the prescription stated above we find an interesting case, the replicated partition function Z'_n is constructed by taking the following combination: uncorrelated normalization and uncorrelated momenta along T^n , correlated momentum on the non-compact

directions and correlated spin structures over the replicas. The result is

$$Z'^c_n = 2\mathcal{N}^2 \left(\frac{1}{2\epsilon n} \right)^{\frac{9-(r+k)}{2}} \left(\prod_{i=1}^s \sum_{w_j} e^{-2\pi\epsilon(w_i R_i)^2} \right)^n \left(\prod_{j=1}^{k-s} \sum_{m_j} e^{-2\pi\epsilon \left(\frac{m_j}{R_j} \right)^2} \right)^n \left(\frac{f_3^8(q_n) - f_4^8(q_n)}{f_1^8(q_n)} \right). \quad (7.3)$$

The corresponding entanglement entropy is:

$$\tilde{S}'^c = \log 2 + \log \mathcal{N}^2 + \frac{9-r}{2} \log 2 + \frac{\pi}{2\epsilon}. \quad (7.4)$$

For the non-BPS brane the normalization constant is

$$\mathcal{N}^2 = \frac{V_{r+1}}{64(2\pi)^{r+1}} \frac{\prod_{j=1}^{k-s} R_j}{\prod_{i=1}^s R_i}.$$

Therefore, the entanglement entropy for a Non-BPS D p -brane on T^k with r non-compact and s compact Neumann directions is:

$$S'^c(r, s) = \log 2 + \log \frac{V_{r+1}}{64(2\pi)^{r+1}} + \log \prod_{j=1}^{k-s} R_j + \log \prod_{i=1}^s \frac{1}{R_i} + \frac{9-r}{2} \log 2 + \frac{\pi}{2\epsilon}. \quad (7.5)$$

Note that T-duality interchanges Neumann and Dirichlet directions. In particular, the radius of compactification is changed as $R \rightarrow \frac{1}{R}$. Starting from (7.5) one obtains the entanglement entropy $S'^c(r, s-1)$ of a D $(p-1)$ brane performing a T-duality on one of the compact Neumann directions. Similarly, the entanglement entropy $S'^c(r, s+1)$ of a D $(p+1)$ -brane is obtained by T-duality on one of the compactified Dirichlet directions. We note that $S^c(r, 0)$ describes the entanglement entropy with all compact directions Dirichlet. $S^c(r, k)$ is the entanglement entropy with all compact Neumann directions. Then, one can verify that

$$S^c(r, 0) > S^c(r, k). \quad (7.6)$$

It is worth pointing out that our main result in eq. (7.5) is compatible with the analysis of [24] which considered the boundary entropy of various compactified configurations, albeit in the bosonic string. In particular, as pointed out in [24] in the context of toroidal compactification, there is a path from the results in terms of boundary entropy to some of the spacetime moduli. The key observation is to view the radii entering in eq. (7.5) as elements of the metric, further introduce a B-field and construct the complex moduli such as $\rho = B_{12} + i\sqrt{\det G}$ on which the entropy will depend. We will not pursue this direction further in this paper but it is clearly interesting and forms part of the more ambitious goal of translating worldsheet left-right entanglement entropy computations into spacetime statements.

8 Conclusions

We have computed the left-right entanglement entropy of string theory Dp branes. Our results can be interpreted as a generalization of the tension of a Dp brane as follows from [18]. This is not surprising, indeed, our previous work in [7] has shown that the left-right entanglement entropy includes the boundary entropy which is precisely related to the tension of a D-brane including the more general case of compactifications [18, 24]. The new aspect which we have addressed in the present manuscript is the presence, in the case of the superstrings Dp branes, of various sectors.

It is clarifying to contrast our computation with that of the string tension given by Polchinski [25]. It has been important for us to consider only a given spin structure and we have chosen the NS one. We have verified that the entanglement entropy vanishes if one includes all the spin structures, just as the overall cylinder diagram vanishes in the tension computation of Polchinski. It is interesting that our computation seems to indicate that the type of entanglement between the left and right sectors depends, indeed, on the type of interaction that is allowed. Namely, as we saw in the text, considering the NS and the RR sectors together leads, due to supersymmetry, to a cancellation. We have argued that our computation is conceptually analogous to [18] which, for the purpose of the D-brane tension, focused exclusively on the interaction with the graviton which resides in the NS-NS sector. Again, choosing a particular spin structure on the world sheet is equivalent to choosing a particular type of spacetime interactions and can thus quantify how entangled two subsystems are due to a particular type of interaction. This is particular of superstring theory where we see clearly the contributions of the gravitational (NS) sector and of the charged (RR) sector.

We have also considered Dp branes on compactified spaces and subsequently studied the left-right entanglement entropy under T-duality. The resulting expression, eq. (7.5), is covariant under T-duality that sends a Dp into a $D(p \pm 1)$.

In this manuscript we have presented the completely un-correlated and completely correlated entanglement entropies. As stated clearly in the conclusions of [17], there is a set of partitions functions in which one can choose to partially correlate the spin structures. It would be important to understand, perhaps based on some other physical criterion, which of the partitions functions is more appropriate. Recall that in the case reported in [17] it was found that, depending of the size of the interval for the entanglement entropy, one was instructed to take the uncorrelated version for ($\ell \rightarrow 0$) while the correlated, taking the product over replicas before summing over spin structures when the interval was close to the whole system size ($\ell \rightarrow L$). Along these lines, we have suggested that the entanglement entropy that most naturally accommodates a thermodynamical interpretation is the one computed using uncorrelated normalization and correlated momentum and spin structure. This is a topic that clearly deserves more attention.

One of the directions that is left somehow unexplored in this manuscript but that constitutes a driving motivation is the interpretation of our worldsheet results in terms of spacetime. The fundamental prototype is the interesting computation of Di Vecchia and collaborators [26] who intuited spacetime properties of D-branes by using worldsheet

computations. The p -dependence of the left-right entanglement entropy of a Dp brane in formulas such as eq. (7.5) seems to be a first incipient step in this direction.

Another interesting direction to explore would be to understand to what extent the entropy computed in this manuscript respects a first-law like relation for the entanglement entropy and the energy of excited states. Such relations have been obtained purely in field theoretic terms [20, 21] but in rather symmetry restricted setups. Equivalently, it would be interesting to understand the structure of the modular Hamiltonian underlying our computation of the reduced density matrix. A connection to a first-law like relation might also lead to a physical criterion as the one used in [17] where the entanglement entropy was expected to be related in a particular way to the thermodynamic entropy as suggested in [27] and later proven in [28].

More speculatively, it would be interesting to explore whether this D-brane worldsheet computation can be used to understand certain field theory entanglement entropy aspects. For example there might be aspects of $\mathcal{N} = 4$ SYM that could be obtained as a particular limit of these configurations of open strings. We hope to explore some of these interesting issues in the future.

Finally, and perhaps more immediately, it would be interesting to place our explicit computations in the more general context of RCFT with supersymmetry. This effort will certainly build upon the beautiful analysis presented in [8] and the background provided in [18]. We hope to soon report some progress in this direction.

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A The f_i functions

We follow [16] to define the functions f_i in terms of the Jacobi Theta functions as:

$$\begin{aligned}
 f_1(q) &= q^{\frac{1}{12}} \prod_{m=1}^{\infty} (1 - q^{2m}) = (2\pi)^{-1/3} \theta'_1(0|\tau)^{1/3}, \\
 f_2(q) &= q^{\frac{1}{12}} \prod_{m=1}^{\infty} (1 + q^{2m}) = (2\pi)^{1/6} \theta_2(0|\tau)^{1/2} \theta'_1(0|\tau)^{-1/6}, \\
 f_3(q) &= q^{-\frac{1}{24}} \prod_{m=1}^{\infty} (1 + q^{2m-1}) = (2\pi)^{1/6} \theta_3(0|\tau)^{1/2} \theta'_1(0|\tau)^{-1/6}, \\
 f_4(q) &= q^{-\frac{1}{24}} \prod_{m=1}^{\infty} (1 - q^{2m-1}) = (2\pi)^{1/6} \theta_4(0|\tau)^{1/2} \theta'_1(0|\tau)^{-1/6},
 \end{aligned} \tag{A.1}$$

where $q = e^{i\pi\tau}$. Under the modular transformation $\tau \rightarrow -1/\tau$, $q \rightarrow \tilde{q} = e^{-i\frac{\pi}{\tau}}$ and the functions transform as

$$\begin{aligned} f_1(\tilde{q}) &= \sqrt{-i\tau} f_1(q), & f_2(\tilde{q}) &= f_4(q) \\ f_3(\tilde{q}) &= f_3(q), & f_4(\tilde{q}) &= f_2(q) \end{aligned} \quad (\text{A.2})$$

B Density matrix for A system

Following [15] we work with the notation $\alpha' = 1$. In order to compute the density matrix, first we define the closed string hamiltonian in the NS-NS sector is

$$H_c = \pi \mathbf{P}^2 + 2\pi \sum_{\mu=2}^9 \left[\sum_{n=1}^{\infty} \left(\alpha_{-n}^{\mu} \alpha_n^{\mu} + \tilde{\alpha}_{-n}^{\mu} \tilde{\alpha}_n^{\mu} \right) + \sum_{r=\frac{1}{2}}^{\infty} \left(\psi_{-r}^{\mu} \psi_r^{\mu} + \tilde{\psi}_{-r}^{\mu} \tilde{\psi}_r^{\mu} \right) \right] - 2\pi. \quad (\text{B.1})$$

Then, for the NS-NS boundary state

$$\langle\langle \rho \rangle\rangle = \sum_{\eta\eta'} \eta\eta' e^{-\epsilon H_c} |Bp, \mathbf{a}, \eta\rangle_{\text{NSNS}} \langle Bp, \mathbf{a}, \eta'| e^{-\epsilon H_c}. \quad (\text{B.2})$$

In order to compute ρ_A we dissect the boundary state and the Hamiltonian in their parts: the momentum contribution and the oscillator modes.

In the definition of ρ_A given in (3.6), the trace in the momentum is denoted as $\text{Tr}_C \rho$. The momentum contribution to the boundary state is expressed as a Fourier transform in the position space localizing the brane in position \mathbf{a} along the Dirichlet directions:

$$|B, \mathbf{a}\rangle = \int \prod_{\nu=0,1,p+3}^9 dk^{\nu} e^{ik^{\nu} a_{\nu}} |\mathbf{k}\rangle. \quad (\text{B.3})$$

Along the Neumann directions the momentum is zero, so the boundary state carries momentum \mathbf{k} along the Dirichlet directions only. To compute the trace we note that for a closed string ground state with momentum \mathbf{p}

$$\langle \mathbf{p} | e^{-\pi\epsilon \mathbf{P}^2} | B, \mathbf{a} \rangle = e^{ip \cdot \mathbf{a}} e^{-\pi\epsilon \mathbf{P}^2}. \quad (\text{B.4})$$

Therefore

$$\text{Tr}_C \langle\langle \rho \rangle\rangle = \text{Tr}_C \left[e^{-\pi\epsilon \mathbf{P}^2} |B, \mathbf{a}\rangle \langle B, \mathbf{a}| e^{-\pi\epsilon \mathbf{P}^2} \right] = \left(\frac{1}{2\epsilon} \right)^{\frac{9-p}{2}}, \quad (\text{B.5})$$

The trace over B , is the trace over the left-oscillator modes. We first consider the bosonic oscillator mode in the definition of ρ . Taking the bosonic part of the string state defined in (3.5) we have:

$$\langle l_L | e^{-\epsilon H_c} | B \rangle = \prod_{\mu=1}^9 \prod_{n=1}^{\infty} e^{-4\pi\epsilon n l_n} \frac{(-1)^{l_n}}{\sqrt{l_n!}} \left(\frac{\alpha_{-n}^{\mu}}{\sqrt{n}} \right)^{l_n} |0\rangle. \quad (\text{B.6})$$

For the fermionic part, we have to consider the contribution to the density matrix of the boundary state with spin structure, that is:

$$\langle l_L | e^{-\pi\epsilon H_c} | B, \eta \rangle = \prod_{\mu=1}^9 \prod_{r=\frac{1}{2}}^{\infty} \left[\delta_{l_r^\mu, 0} + i\eta e^{-4\pi\epsilon r} S_{\mu\mu} \delta_{l_r^\mu, 1} \psi_{-r}^\mu \right] |0\rangle. \quad (\text{B.7})$$

Defining $q = e^{-4\pi\epsilon}$ we have

$$\text{Tr}_B \langle \langle \rho \rangle \rangle = q^{-1} \prod_{\mu} \left[\prod_n \sum_{l_n^\mu} q^{2nl_n^\mu} \frac{1}{l_n^\mu!} \left(\frac{\alpha_{-n}^\mu}{\sqrt{n}} \right)^{l_n^\mu} \otimes \left(\frac{\alpha_n^\mu}{\sqrt{n}} \right)^{l_n^\mu} \right] \left[\prod_r (1 \otimes 1 + \eta\eta' q^{2r} \psi_{-r}^\mu \otimes \psi_r^\mu) \right] |0\rangle \otimes \langle 0|, \quad (\text{B.8})$$

where we have included in the bosonic part the contrubution of the zero-point energy. The notation for the tensor product is that for A y B operators, $(A \otimes B)(|0\rangle \otimes \langle 0|) = A|0\rangle \otimes \langle 0|B$. Finally, the desired trace is

$$\begin{aligned} \langle \langle \rho_A \rangle \rangle &= \langle \langle \text{Tr}_B(\text{Tr}_C \rho) \rangle \rangle = \mathcal{N}^2 \left(\frac{1}{2\epsilon} \right)^{\frac{9-p}{2}} q^{-1} \prod_{\mu} \prod_n \sum_{l_n^\mu} q^{2nl_n^\mu} \frac{1}{l_n^\mu!} \left(\frac{\alpha_{-n}^\mu}{\sqrt{n}} \right)^{l_n^\mu} \otimes \left(\frac{\alpha_n^\mu}{\sqrt{n}} \right)^{l_n^\mu} \\ &\times \sum_{\eta\eta'} \left[\prod_{\mu} \prod_r (1 \otimes 1 + \eta\eta' q^{2r} \psi_{-r}^\mu \otimes \psi_r^\mu) \right] |0\rangle \otimes \langle 0|. \end{aligned} \quad (\text{B.9})$$

For ρ_{AC} we do not sum over the momentum, then we have:

$$\begin{aligned} \langle \langle \rho_{AC} \rangle \rangle &= \text{Tr}_B \langle \langle \rho_{AC} \rangle \rangle = \mathcal{N}^2 \int \prod_{\nu} \int \prod_{\nu'} dk^\nu dk'^{\nu'} e^{i\mathbf{a} \cdot (\mathbf{k} - \mathbf{k}')} e^{-\pi(\mathbf{k}^2 - \mathbf{k}'^2)} q^{-1} \\ &\times \prod_{\mu} \left[\prod_n \sum_{l_n^\mu} q^{2nl_n^\mu} \frac{1}{l_n^\mu!} \left(\frac{\alpha_{-n}^\mu}{\sqrt{n}} \right)^{l_n^\mu} \otimes \left(\frac{\alpha_n^\mu}{\sqrt{n}} \right)^{l_n^\mu} \right] \left[\prod_r (1 \otimes 1 + \eta\eta' q^{2r} \psi_{-r}^\mu \otimes \psi_r^\mu) \right] |0, \mathbf{k}\rangle \otimes \langle 0, \mathbf{k}|. \end{aligned} \quad (\text{B.10})$$

C Supersymmetric branes do not get entangled!

This appendix has a two-fold motivation. First we want to display explicit aspects of the calculation of the entanglement entropy in the context of D-brane and second we want to highlight the fact that for supersymmetric branes the entanglement vanishes identically when sum over all spin structures.

The vanishing of the entanglement is akin to the vanishing one-loop diagram in string theory between two D-branes and we learn that to get a non-trivial entanglement one needs to consider sectors of a given spin structure.

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References

- [1] A. Strominger and C. Vafa, *Microscopic origin of the Bekenstein-Hawking entropy*, *Phys. Lett. B* **379** (1996) 99 [[hep-th/9601029](#)] [[INSPIRE](#)].
- [2] C.G. Callan and J.M. Maldacena, *D-brane approach to black hole quantum mechanics*, *Nucl. Phys. B* **472** (1996) 591 [[hep-th/9602043](#)] [[INSPIRE](#)].
- [3] L. Bombelli, R.K. Koul, J. Lee and R.D. Sorkin, *A Quantum Source of Entropy for Black Holes*, *Phys. Rev. D* **34** (1986) 373 [[INSPIRE](#)].
- [4] M. Srednicki, *Entropy and area*, *Phys. Rev. Lett.* **71** (1993) 666 [[hep-th/9303048](#)] [[INSPIRE](#)].
- [5] L. Taddia, J.C. Xavier, F.C. Alcaraz and G. Sierra, *Entanglement entropies in conformal systems with boundaries*, *Phys. Rev. B* **88** (2013) 075112 [[arXiv:1302.6222](#)].
- [6] X.-L. Qi, H. Katsura and A.W.W. Ludwig, *General Relationship between the Entanglement Spectrum and the Edge State Spectrum of Topological Quantum States*, *Phys. Rev. Lett.* **108** (2012) 196402 [[arXiv:1103.5437](#)].
- [7] L.A. Pando Zayas and N. Quiroz, *Left-Right Entanglement Entropy of Boundary States*, *JHEP* **01** (2015) 110 [[arXiv:1407.7057](#)] [[INSPIRE](#)].
- [8] D. Das and S. Datta, *Universal features of left-right entanglement entropy*, *Phys. Rev. Lett.* **115** (2015) 131602 [[arXiv:1504.02475](#)] [[INSPIRE](#)].
- [9] E.M. Brehm and I. Brunner, *Entanglement entropy through conformal interfaces in the 2D Ising model*, *JHEP* **09** (2015) 080 [[arXiv:1505.02647](#)] [[INSPIRE](#)].
- [10] H.J. Schnitzer, *Left-Right Entanglement Entropy, D-branes and Level-rank duality*, [[arXiv:1505.07070](#)] [[INSPIRE](#)].
- [11] E.M. Brehm, I. Brunner, D. Jaud and C. Schmidt-Colinet, *Entanglement and topological interfaces*, *Fortsch. Phys.* **64** (2016) 516 [[arXiv:1512.05945](#)] [[INSPIRE](#)].
- [12] M. Gutperle and J.D. Miller, *A note on entanglement entropy for topological interfaces in RCFTs*, *JHEP* **04** (2016) 176 [[arXiv:1512.07241](#)] [[INSPIRE](#)].
- [13] J.M. Maldacena, *Eternal black holes in anti-de Sitter*, *JHEP* **04** (2003) 021 [[hep-th/0106112](#)] [[INSPIRE](#)].
- [14] T. Takayanagi and T. Ugajin, *Measuring Black Hole Formations by Entanglement Entropy via Coarse-Graining*, *JHEP* **11** (2010) 054 [[arXiv:1008.3439](#)] [[INSPIRE](#)].
- [15] M.R. Gaberdiel, *Lectures on nonBPS Dirichlet branes*, *Class. Quant. Grav.* **17** (2000) 3483 [[hep-th/0005029](#)] [[INSPIRE](#)].
- [16] M.R. Gaberdiel and B. Stefanski Jr., *Dirichlet branes on orbifolds*, *Nucl. Phys. B* **578** (2000) 58 [[hep-th/9910109](#)] [[INSPIRE](#)].
- [17] S.F. Lokhande and S. Mukhi, *Modular invariance and entanglement entropy*, *JHEP* **06** (2015) 106 [[arXiv:1504.01921](#)] [[INSPIRE](#)].
- [18] J.A. Harvey, S. Kachru, G.W. Moore and E. Silverstein, *Tension is dimension*, *JHEP* **03** (2000) 001 [[hep-th/9909072](#)] [[INSPIRE](#)].
- [19] I. Affleck and A.W.W. Ludwig, *Universal noninteger ‘ground state degeneracy’ in critical quantum systems*, *Phys. Rev. Lett.* **67** (1991) 161 [[INSPIRE](#)].
- [20] D.D. Blanco, H. Casini, L.-Y. Hung and R.C. Myers, *Relative Entropy and Holography*, *JHEP* **08** (2013) 060 [[arXiv:1305.3182](#)] [[INSPIRE](#)].

- [21] G. Wong, I. Klich, L.A. Pando Zayas and D. Vaman, *Entanglement Temperature and Entanglement Entropy of Excited States*, *JHEP* **12** (2013) 020 [[arXiv:1305.3291](#)] [[INSPIRE](#)].
- [22] I.R. Klebanov, D. Kutasov and A. Murugan, *Entanglement as a probe of confinement*, *Nucl. Phys. B* **796** (2008) 274 [[arXiv:0709.2140](#)] [[INSPIRE](#)].
- [23] I. Bah, A. Faraggi, L.A. Pando Zayas and C.A. Terrero-Escalante, *Holographic entanglement entropy and phase transitions at finite temperature*, *Int. J. Mod. Phys. A* **24** (2009) 2703 [[arXiv:0710.5483](#)] [[INSPIRE](#)].
- [24] S. Elitzur, E. Rabinovici and G. Sarkissian, *On least action D-branes*, *Nucl. Phys. B* **541** (1999) 246 [[hep-th/9807161](#)] [[INSPIRE](#)].
- [25] J. Polchinski, *Dirichlet Branes and Ramond-Ramond charges*, *Phys. Rev. Lett.* **75** (1995) 4724 [[hep-th/9510017](#)] [[INSPIRE](#)].
- [26] P. Di Vecchia, M. Frau, I. Pesando, S. Sciuto, A. Lerda and R. Russo, *Classical p-branes from boundary state*, *Nucl. Phys. B* **507** (1997) 259 [[hep-th/9707068](#)] [[INSPIRE](#)].
- [27] T. Azeyanagi, T. Nishioka and T. Takayanagi, *Near Extremal Black Hole Entropy as Entanglement Entropy via AdS_2/CFT_1* , *Phys. Rev. D* **77** (2008) 064005 [[arXiv:0710.2956](#)] [[INSPIRE](#)].
- [28] B. Chen and J.-q. Wu, *Universal relation between thermal entropy and entanglement entropy in conformal field theories*, *Phys. Rev. D* **91** (2015) 086012 [[arXiv:1412.0761](#)] [[INSPIRE](#)].